Guided Notes
Section 10.2

Calculus with parametric Equations

Suppose f and g are differentiable functions and we want to find the tangent line on the point of a parametric curve with $x = f(t)$ and $y = g(t)$. y is also a differentiable function of x.

Then $\boxed{\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}}$ if $\dfrac{dx}{dt} \neq 0$

The second derivative

$$\boxed{\frac{d^2y}{dx^2}} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right) = \frac{\frac{d}{dx}\left(\frac{dy}{dt}\right)}{\frac{dx}{dt}} = \boxed{\frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}}$$

note $\boxed{\dfrac{d^2y}{dx^2} \neq \dfrac{\frac{d^2y}{dt^2}}{\frac{d^2x}{dt^2}}}$

Another way to look at this

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(y') = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}} = \boxed{\frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}}$$

using the first derivative relationship

Example 1

Find the slope of the tangent and normal lines at point $P(t)$ on the curve having parametric equations $x = 2t$ and $y = t^2 - 1$ where $-1 \le t \le 2$

tangent line

$$m = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \qquad m = \frac{2t}{2} = t \qquad \boxed{m_{tan} = t}$$

normal line A line which is perpendicular to a tangent line

$$\boxed{m = -\frac{1}{t}} \qquad t \neq 0$$

Example 2

If C has parametric equations $x = t^3 - 3t$ and $y = t^2 - 5t - 1$ where $t \in \mathbb{R}$, find an equation of a tangent line to C when $t = 2$. For what values is the tangent horizontal? vertical?

$$m = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t-5}{3t^2-3}$$

$$x = 2^3 - 3(2) = 2$$
$$y = 2^2 - 5(2) - 1 = -7$$
$$(2, -7)$$

when $t = 2$ $\quad m = \frac{2(2)-5}{3(2)^2-3} = -\frac{1}{9}$

$$\boxed{y + 7 = -\frac{1}{9}(x - 2)}$$

$\boxed{\text{horizontal}}$
$\boxed{m=0}$

$$\frac{2t-5}{3t^2-3}=0 \qquad 2t-5=0$$
$$2t=5 \quad t=\frac{5}{2}$$

$\boxed{\text{vertical}}$
$\boxed{m \text{ is undefined}}$

$$3t^2-3=0$$
$$3t^2=3 \qquad \boxed{t=\pm 1}$$
$$t^2=1$$

$\boxed{\text{Example 3}}$

Suppose C has parametric equations
$x=e^{-t}$ $\quad t\in\mathbb{R}$
$y=e^{2t}$

$\boxed{\text{a) Find } \frac{d^2y}{dx^2} \text{ using the parametric equations directly}}$

$$\boxed{\frac{d^2y}{dx^2}}=\frac{dy'}{dx}=\frac{\frac{dy'}{dt}}{\frac{dx}{dt}}=\frac{\frac{d}{dt}(-2e^{3t})}{-e^{-t}}=\frac{-6e^{3t}}{e^{-t}}$$
$$=\boxed{6e^{4t}}$$

$x=e^{-t}$
$\frac{dx}{dt}=-e^{-t}$

$y=e^{2t}$
$\frac{dy}{dt}=2e^{2t}$

$$\text{so } \boxed{\frac{dy}{dx}}=\frac{2e^{2t}}{-e^{-t}}=\boxed{-2e^{3t}}$$

$k(x) = y$

(b) Determine a function k which has the same graph as c. Find the second derivative of k and compare your answers.

$x = e^{-t}$ $\quad$ $y = e^{2t}$

$x = \dfrac{1}{e^t}$ $\quad$ $\boxed{y = \left(\dfrac{1}{x}\right)^2 = \dfrac{1}{x^2}}$ $\quad$ $\boxed{x > 0}$

$e^t = \dfrac{1}{x}$

note that $e^{-t} > 0$ for all t

$y = x^{-2}$

$y' = -2x^{-3}$

$y'' = 6x^{-4} = \boxed{\dfrac{6}{x^4}} = \boxed{6e^{4t}}$

(c) Discuss the concavity of c

$y'' = \dfrac{6}{x^4}$ $\quad$ $6 \neq 0$ $\quad$ $x^4 \neq 0$ $\quad$ $x \neq 0$

y'' $\quad$ $+$ $\quad$ | $\quad$ $+$

0

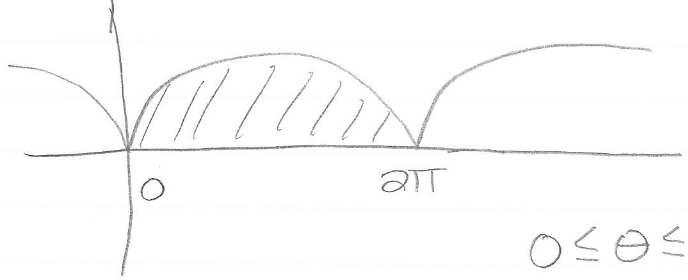
In terms of t

$\dfrac{d^2y}{dx^2} = 6e^{4t} > 0$ for all t

so the curve is always concave up.

Example 4

Find the area under one arc of the cycloid $x = r(\theta - \sin\theta)$
$y = r(1 - \cos\theta)$



$0 \le \theta \le 2\pi$

Recall

$$A = \int_a^b F(x)\,dx$$

we need to convert this

$$A = \int_a^b F(x)\,dx = \int_a^b y\,dx = \int_\alpha^\beta g(t)\,f'(t)\,dt$$

$x = r(\theta - \sin\theta)$ $\qquad$ $y = r(1 - \cos\theta)$

$\frac{dx}{d\theta} = r(1 - \cos\theta)$

$dx = r(1 - \cos\theta)\,d\theta$

for the first cycloid

$$A = \int_0^{2\pi r} y\,dx = \int_0^{2\pi} r(1-\cos\theta)\,r(1-\cos\theta)\,d\theta$$

$$= \int_0^{2\pi} r^2(1-\cos\theta)^2\,d\theta = r^2\int_0^{2\pi}(1 - 2\cos\theta + \cos^2\theta)\,d\theta$$

$$= r^2 \int_0^{2\pi} \left[1 - 2\cos\theta + \tfrac{1}{2}(1 + \cos 2\theta)\right] d\theta$$

$$= r^2 \left[\theta - 2\sin\theta + \tfrac{1}{2}\left(\theta + \tfrac{1}{2}\sin 2\theta\right) \right]\Big|_0^{2\pi}$$

$$= r^2 \left[\tfrac{3}{2}\theta - 2\sin\theta + \tfrac{1}{4}\sin 2\theta \Big|_0^{2\pi} \right] = \boxed{3\pi r^2}$$

Arc Length

If a curve C is described by parametric equations $x = f(t)$ and $y = g(t)$, $\alpha \leq t \leq \beta$ where f' and g' are continuous on $[\alpha, \beta]$ and C is traversed exactly once as t increases from α to β, then the length of C is

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Example 5 Let $x = \cos t$ and $y = \sin t$
$0 \leq t \leq 2\pi$

Find the arc length.

$$\frac{dx}{dt} = -\sin t \qquad L = \int_0^{2\pi} \left((-\sin t)^2 + (\cos t)^2\right) dt$$

$$\frac{dy}{dt} = \cos t$$

$$L = \int_0^{2\pi} dt = \boxed{2\pi}$$

Example 6 (# 42)

Find the arc length when $x = e^t - t$
$y = 4e^{\frac{1}{2}t}$, $0 \le t \le 2$

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$x = e^t - t$ $\quad$ $y = 4e^{t/2}$

$\frac{dx}{dt} = e^t - 1$ $\quad$ $\frac{dy}{dt} = 4\left(\frac{1}{2}\right)e^{t/2} = 2e^{t/2}$

$$L = \int_0^2 \sqrt{(e^t - 1)^2 + (2e^{t/2})^2}\, dt$$

$$L = \int_0^2 \sqrt{e^{2t} - 2e^t + 1 + 4e^t}\, dt$$

$$L = \int_0^2 \sqrt{e^{2t} + 2e^t + 1}\, dt = \int_0^2 \sqrt{(e^t + 1)^2}\, dt$$

$$L = \int_0^2 (e^t + 1)\, dt = e^t + t \Big|_0^2 =$$

$$= e^2 + 2 - e^0 - 0 = e^2 + 2 - 1 =$$

$$\boxed{e^2 + 1}$$

Surface Area with parametric equations

$$\boxed{S = \int_{\alpha}^{\beta} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt}$$

Recall this was adapted from

$$S = \int 2\pi y\, ds \quad \text{and} \quad S = \int 2\pi x\, ds$$

for parametrics $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

Example 7

Show that the surface area of rotating a sphere of radius r about the x-axis is $4\pi r^2$.

$x = r\cos t$
$y = r\sin t$
$0 \le t \le \pi$

about x $\quad \int_{\alpha}^{\beta} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$$S = \int_0^{\pi} 2\pi \underbrace{y}_{r\sin t} \sqrt{(-r\sin t)^2 + (r\cos t)^2}\, dt$$

$$S = 2\pi \int_0^{\pi} r\sin t \sqrt{r^2(\sin^2 t + \cos^2 t)}\, dt =$$

$$S = 2\pi r^2 \int_0^{\pi} \sin t\, dt = 2\pi r^2(-\cos t)\Big|_0^{\pi} = \boxed{4\pi r^2}$$

Surface area

#62 $x = 2t^2 + \frac{1}{t}$ $\quad y = 8\sqrt{t}$ $\quad 1 \le t \le 3$

$2t^2 + t^{-1}$ $\quad$ about the x-axis.

$$S = 2\pi \int_1^3 y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \qquad \frac{dx}{dt} = 4t - t^{-2}$$

$$\frac{dy}{dt} = \frac{1}{2}(8)t^{-1/2} = 4t^{-1/2}$$

$$S = 2\pi \int_1^3 8t^{1/2} \sqrt{\left(4t - \frac{1}{t^2}\right)^2 + \left(\frac{4}{t^{1/2}}\right)^2}\, dt$$

$$S = 2\pi \int_1^3 8t^{1/2} \sqrt{16t^2 - 2\left(4t \cdot \frac{1}{t^2}\right) + \left(\frac{1}{t^2}\right)^2 + \left(\frac{4}{t^{1/2}}\right)^2}\, dt$$

$$S = 2\pi \int_1^3 8t^{1/2} \sqrt{16t^2 - \frac{8}{t} + \frac{1}{t^4} + \frac{16}{t}}\, dt$$

$$S = 2\pi \int_1^3 8t^{1/2} \sqrt{16t^2 + \frac{8}{t} + \frac{1}{t^4}}\, dt$$

$$S = 2\pi \int_1^3 8t^{1/2} \sqrt{\frac{16t^6 + 8t^3 + 1}{t^4}}\, dt$$

$$S = 2\pi \int_1^3 8t^{1/2} \left(\frac{1}{t^2}\right) \left[\sqrt{\underbrace{16t^6 + 8t^3 + 1}_{(4t^3+1)^2}}\right] dt =$$

Surface area

#66 $x = e^t - t \qquad y = 4e^{t/2} \qquad 0 \le t \le 1$
about the y-axis

$$S = \int_0^1 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

$\frac{dx}{dt} = e^t - 1$

$\frac{dy}{dt} = 2e^{t/2}$

$$S = \int_0^1 2\pi (e^t - t) \sqrt{(e^t-1)^2 + (2e^{t/2})^2}\, dt$$

$$S = 2\pi \int_0^1 (e^t - t) \sqrt{(e^{2t} - 2e^t + 1 + 4e^t)}\, dt$$

$$S = 2\pi \int_0^1 (e^t - t) \sqrt{(e^{2t} + 2e^t + 1)}\, dt$$

$$S = 2\pi \int_0^1 (e^t - t) \sqrt{(e^t + 1)^2}\, dt$$

	D	I
+	$-t$	e^t
−	-1	e^t
+	0	e^t

$-te^t + e^t$

$$S = 2\pi \int_0^1 (e^t - t)(e^t + 1)\, dt$$

$$S = 2\pi \int_0^1 (e^{2t} + e^t - \underbrace{te^t}_{\text{parts}} - t)\, dt$$

$$S = 2\pi \left[\tfrac{1}{2} e^{2t} + 2e^t - te^t - \frac{t^2}{2} \right]_0^1 = \boxed{\pi (e^2 + 2e - 6)}$$

$$= 16\pi \int_1^3 t^{3/2}\left(4t^{3}+1\right)dt \qquad \frac{6}{2}$$

$$= 16\pi \int_1^3 \left(4t^{3/2} + t^{-3/2}\right)dt \qquad \frac{-2}{\sqrt{t}}$$

$$= 16\pi \left[4\left(\frac{2}{5}\right)t^{5/2} + \frac{-2}{1}t^{-1/2} \Big|_1^3 \right] =$$

$$= 16\pi \left[\left(\frac{8}{5}(3)^{5/2} - 2(3)^{-1/2}\right) - \left(\frac{8}{5} - \frac{2}{1}\right) \right]$$

$$= 16\pi \left[\frac{8}{5}(3)^{2}(3)^{1/2} - 2\frac{1}{\sqrt{3}} - \frac{8}{5} + 2 \right]$$

$$= 16\pi \left[\frac{8(9)\sqrt{3}}{5} - \frac{2}{\sqrt{3}} - \frac{8}{5} + 2 \right] \qquad \frac{-2}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{-2\sqrt{3}}{3}$$

$$= 16\pi \left[\frac{8(9)}{5}\sqrt{3} - \frac{2\sqrt{3}}{3} - \frac{8}{5} + \frac{2}{1} \right]$$

$$= 16\pi \left[\frac{8(9)(3)\sqrt{3} - 2\sqrt{3}(5) - 8(3) + 2(15)}{15} \right]$$

$$= 16\pi \left[\frac{216\sqrt{3} - 10\sqrt{3} - 24 + 30}{15} \right] = \frac{16\pi}{15}\left[206\sqrt{3} + 6\right]$$

$$= \boxed{\frac{32\pi}{15}\left[103\sqrt{3} + 6\right]}$$